Algebraic Geometry Mid Term

February 22 2024

This exam is of **30 marks** and is **3 hours long**. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

1. Let $f \in k[X, Y]$.

$$f = a + bX + cY + dX^2 + eXY + fY^2 + gX^3 + \cdots$$

and let $C = V(f)$ in $\mathbb{A}_k^2$. Write down conditions under which

1. $P = (0, 0) \in C$ (1)
2. The tangent line to C at P is $y = 0$. (2)
3. P is an inflection point of C with $y = 0$ as the tangent line (2)
4. P is a singular point of C. (2)

2. Let k be any field.

- Prove that any algebraic set in $\mathbb{A}_k^1$ is either the whole of $\mathbb{A}_k^1$ or a finite set of points. (2)
- Let f and g be irreducible elements of $k[X, Y]$ such that f and g are not multiples of each other. Prove that $V(f, g)$ is finite. (3)
- Prove that any proper algebraic subset of $\mathbb{A}_k^2$ is a finite union of points and curves. (2)

3. Let $C : Y^3 = X^4 + X^3 \subset \mathbb{A}^2$.

- Show that $(X, Y) \to X/Y$ defines a rational map $\phi : C \dashrightarrow \mathbb{P}^1$ and its inverse is a polynomial $\psi : \mathbb{A}^1 \to C$. (2)
- Show that ψ restricts to an isomorphism (2)

$$\mathbb{A}^1 - \{3\, points\} \simeq C - \{(0,0)\}$$

- Are these two curves birational? (1)

4. Recall that the *Segre embedding* is the map

$$\sigma_{m,n} : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow S_{n,m} \subset \mathbb{P}^{(m+1)(n+1)-1}$$

obtained by sending $X_i, Y_j \to X_i Y_j$

- Show that the image of $\sigma_{1,1}$ is the Quadric surface

$$S_{1,1} = Q : V(X_0X_3 - X_1X_2) \subset \mathbb{P}^2$$

(2)

- Show that you can find disjoint lines in Q and they have infinitely many lines in Q transversal to them. (2)
- Use that to show that $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$. (2)
- Show that if S is a cubic surface in $\mathbb{P}^3$ with 4 disjoint lines lying on a quadric surface Q then $Q \subset S$. (5)